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Full $O(\alpha_S)$ Evaluation of $\mathbf{b \to s\gamma}$ Transverse Momentum Distribution

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Abstract

The full $O(\alpha_S)$ transverse momentum distribution for the $b \to s\gamma$ decay is computed. Results are presented in analytic form. An improved expression for the coefficient function taking into account subleading operators is given and an exact expression for the remainder function associated with the leading operator $\hat{\mathcal{O}}_7$ is also derived.

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1 Introduction

Perturbation theory, i.e. the expansion in powers of α_S, has been applied to describe decays of the beauty quark since its discovery. While the expansion parameter $\alpha_S(m_B) \sim 0.21$, being reasonably small, allows one to have confidence in the computations, it is difficult to directly compare the perturbative approach with the experimental data. As is well known, decay rates do not make good quantities to be compared with the data, because they are proportional to the fifth power of the beauty quark mass, a poorly known parameter,

$$\Gamma \propto m_b^5 \tag{1}$$

and because they involve in principle unknown CKM matrix elements such as $V_{cb},\ V_{ub},\ V_{ts}$, etc.. By taking ratios of different widths, one can cancel the m_b^5 dependence in the observables, and, eventually, also the dependence on the CKM matrix elements. A rather good theoretical quantity is represented, for instance, by the semileptonic branching ratio:

$$B_{SL} = \frac{\Gamma_{SL}}{\Gamma_{TOT}}, \tag{2}$$

which turns out to be marginally in agreement with present data [1]. Inclusive quantities, $B_{inclusive}$, such as (2), have a perturbative series that involves numerical coefficients c_n of the form:

$$B_{inclusive} \ = \ \sum_{n=0}^{\infty} c_n \ \alpha_S^n(m_B). \tag{3}$$

In less inclusive quantities, additional dynamical effects appear, due to the kinematical restrictions on the final particles, and the use of perturbation theory is, in general, less justified. In semi-inclusive quantities, $B_{semi-inclusive}$, such as threshold and transverse momentum p_t distributions, the perturbative series contains large infrared logarithms in addition to the coefficients c_n; they may be expanded as a perturbative series of the form:

$$B_{semi-inclusive} \ = \ \sum_{n=0}^{\infty} \sum_{k=0}^{2n} c_{n,k} \ \alpha_S^n \ \log^k x, \tag{4}$$

where x represents the characteristic scale of the process as the energy or the transverse momentum. Resummation of such enhanced terms to any order in α_S can be performed in various approximations. The simplest one, the leading logarithmic approximation, involves picking up only the terms having two powers of the logarithm for each power of the coupling, i.e. $k = 2n$. In the double-logarithmic approximation each parton is dressed with a cloud of soft and collinear gluons. Further, more refined approximations involve smaller numbers of logarithms for each power of α_S, i.e. $k = 2n-1, 2n-2, \ldots$. In the last years, considerable effort has been devoted to the study of various spectra in B decays in the endpoint region, in the framework of resummed perturbation theory.
In order to verify the ability of the resummed perturbation theory to describe B decays in a different dynamical situation, we considered, in a previous note [2], p_t-distributions describing that of the s quark with respect to the photon direction, in the b rest frame.
In this work, [2], the following issues have been considered: the resummed p_t-distribution in the $b \to s\gamma$ decay is evaluated and both perturbative and non-perturbative sources of transverse momentum contributions discussed. The general theoretical framework for the evaluation of the corresponding matrix element defined and the strategy to evaluate leading and next-to-leading perturbative contributions is outlined, by introducing a method to treat the radiative corrections and their summation in a improved perturbative formula. The comparison of the transverse momentum distribution singularity structure with the more widely-known threshold case is also presented.
The chosen quantity manifests a clear advantage from a phenomenological point of view since, as discussed in [2], it depends only on the photon momentum in the process. Thanks to the straightforward

and direct kinematics, the transverse momentum turns out to be a particularly simple variable to use to discuss the singularity structure of the perturbative expansion. The case of a possible effective theory within which to factorize these singularities can, for the transverse momentum, be considered as well.

The general formula representing the complete perturbative expression for a the resummed distribution is given by the formula

$$D(x) = K(\alpha_S)\Sigma(x;\alpha_S) + R(x;\alpha_S). \tag{5}$$

The results, already presented in [2], did concern the universal process-independent function $\Sigma(x;\alpha_S)$, resumming the infrared logarithms in exponentiated form.

Here the general perturbative expression for the whole distribution will be concisely recalled and the new entries represented by the coefficient function $K(\alpha_S)$ and by the remainder function $R(x)$ will be evaluated. Both $K(\alpha_S)$ and $R(x)$ are process-dependent and require an explicit evaluation of Feynman diagrams.

Resummation of large infrared logarithms in b decays have been studied in great detail in recent years. This scheme is justified by the fact that the double logarithm appearing to order α_S can become rather large (with respect to 1 coming from the tree level):

$$-\frac{\alpha_S C_F}{4\pi}\log^2\frac{p_t^2}{m_b^2} \sim -0.7 \tag{6}$$

if we push the transverse momentum to such small values as $p_t \sim \Lambda_{QCD} = 300$ MeV. The single logarithm can also become rather large, having a large numerical coefficient:

$$-\frac{5\alpha_S C_F}{4\pi}\log\frac{p_t^2}{m_b^2} \sim 0.6. \tag{7}$$

The purpose of resumming classes of such terms therefore seems quite justified. If we consider running coupling effects, i.e. if the (frozen) coupling evaluated at the hard scale $Q = m_B = 5.2$ GeV is replaced by the coupling evaluated at the gluon transverse momentum,

$$\alpha_S(m_b) \to \alpha_S(p_t) = 0.45 \qquad \text{for} \qquad p_t = 1 \text{ GeV}, \tag{8}$$

the logarithmic terms have sizes of order:

$$-\frac{\alpha_S(p_t) C_F}{4\pi}\log^2\frac{p_t^2}{m_b^2} \sim -0.5 \tag{9}$$

and

$$-\frac{5\alpha_S C_F}{4\pi}\log\frac{p_t^2}{m_b^2} \sim 0.8. \tag{10}$$

The main difference with respect to resummation in Z^0 decays is a hard scale smaller by over an order of magnitude, i.e. a coupling larger by a factor 2 and infrared logarithms smaller by a factor 3.

2 The effective hamiltonian for the decay $b \to s\gamma$

The decay $b \to s\gamma$ is loop-mediated in the Standard Model and offers stringent tests of the latter as well as a way to extract CKM matrix elements. The relevant diagrams involve a loop with a virtual W and an up-type quark (u, c or t); the external photon can be emitted from the internal lines and from the external lines of the b or s quark (see fig. 1). QCD radiative corrections are affected by large logarithms of the form

$$\alpha_S^n \, \log^k\frac{m_W}{m_b} \qquad \text{with} \quad 0 \le k \le n \tag{11}$$

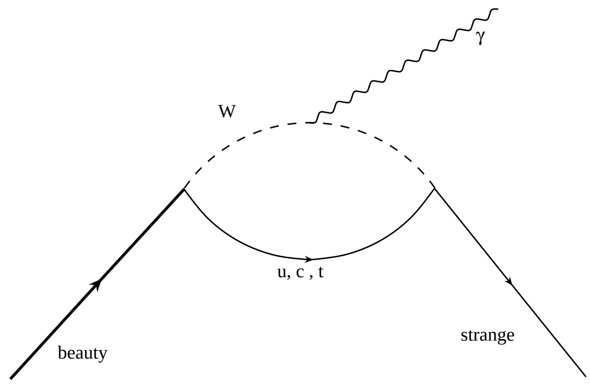

Figure 1: Vertex for $b \to s\gamma$ in the Standard Model

as well as logarithms of m_t/m_W. Since the energies involved in the process are much smaller than the W or t mass, it is possible to integrate out these fields by means of an operator product expansion and write an effective low-energy hamiltonian of the form:

$$\mathcal{H}_{eff}(x) = \frac{G_F}{\sqrt{2}}\ V_{ts}^* V_{tb}\ \sum_{j=1}^{8} C_j(\mu_b)\ \hat{\mathcal{O}}_j(x;\mu_b). \tag{12}$$

With a factorization scale $\mu_b = O(m_b)$, the long-distance effects — both perturbative and non-perturbative — are factorized in the matrix elements of the operators $\hat{\mathcal{O}}_j$, while the short-distance effects are contained in the coefficient functions $C_j(\mu_b)$, calculable in perturbation theory. In particular, the large logarithms in (11) are included into the coefficient functions and can be resummed with standard renormalization group techniques.

A suitable basis for the operators $\hat{\mathcal{O}}_j$ is given by six four-quark operators, $\hat{\mathcal{O}}_1$-$\hat{\mathcal{O}}_6$ and by the penguin operators $\hat{\mathcal{O}}_7$, $\hat{\mathcal{O}}_8$ [4, 5]:

$$\begin{aligned}
\hat{\mathcal{O}}_1 &= (\overline{c}_{L,\beta}\gamma_\mu b_{L,\alpha})(\overline{s}_{L,\alpha}\gamma_\mu c_{L,\beta}) \\
\hat{\mathcal{O}}_2 &= (\overline{c}_{L,\alpha}\gamma_\mu b_{L,\alpha})(\overline{s}_{L,\beta}\gamma_\mu c_{L,\beta}) \\
\hat{\mathcal{O}}_3 &= (\overline{s}_{L,\alpha}\gamma_\mu b_{L,\alpha})(\sum_q \overline{q}_{L,\beta}\gamma_\mu q_{L,\beta}) \\
\hat{\mathcal{O}}_4 &= (\overline{s}_{L,\alpha}\gamma_\mu b_{L,\beta})(\sum_q \overline{q}_{L,\beta}\gamma_\mu q_{L,\alpha}) \\
\hat{\mathcal{O}}_5 &= (\overline{s}_{L,\alpha}\gamma_\mu b_{L,\alpha})(\sum_q \overline{q}_{R,\beta}\gamma_\mu q_{R,\beta}) \\
\hat{\mathcal{O}}_6 &= (\overline{s}_{L,\alpha}\gamma_\mu b_{L,\beta})(\sum_q \overline{q}_{R,\beta}\gamma_\mu q_{R,\alpha}) \\
\hat{\mathcal{O}}_7 &= \frac{e}{16\pi^2} m_{b,\overline{MS}}(\mu_b)\overline{s}_{L,\alpha}\sigma^{\mu\nu} b_{R,\alpha} F_{\mu\nu} \\
\hat{\mathcal{O}}_8 &= \frac{g}{16\pi^2} m_{b,\overline{MS}}(\mu_b)\overline{s}_{L,\alpha}\sigma^{\mu\nu} T^a_{\alpha\beta} b_{R,\alpha} G^a_{\mu\nu},
\end{aligned} \tag{13}$$

where $m_{b,\overline{MS}}(\mu_b)$ is the b mass in the $\overline{MS}$ scheme, evaluated at μ_b and $q = u, d, s, c$ or b.

The dimension of these operators is six: higher-dimension operators have coefficients suppressed by inverse powers of the masses of the integrated particles (t and W) and do not contribute in first

approximation.
The calculation of the QCD corrections to the coefficients functions has been carried out in [6] with leading logarithmic accuracy and in [7] at next-to-leading level in the $\overline{MS}$ scheme.
Let us now consider the evaluation of the matrix elements of the effective hamiltonian between quark states. Only the magnetic penguin operator $\hat{\mathcal{O}}_7$ contributes in lowest order with a rate:[1]

$$\Gamma_0 \simeq \frac{\alpha_{em}}{\pi} \frac{G_F^2\, m_b^3 m^2_{b,\overline{MS}}\,(m_b)\ |V_{tb}V_{ts}^*|^2}{32\pi^3} C_7^2\,(\mu_b)\,, \tag{14}$$

where m_b is the pole mass of the b quark.
Radiative QCD corrections involve gluon brehmsstrahlung. The operator $\hat{\mathcal{O}}_7$ is affected by infrared singularities for the emission of a soft or a collinear gluon; the remaining operators $\hat{\mathcal{O}}_1$-$\hat{\mathcal{O}}_6$ have infrared-finite matrix elements. This implies that QCD corrections to the operator $\hat{\mathcal{O}}_7$ only are logarithmically enhanced for $p_t \ll m_b$ [2]. We will then consider at first only the operator $\hat{\mathcal{O}}_7$.

3 Transverse momentum distribution in $b \to s\gamma$

The process we are dealing with has a very simple kinematics: in lowest order it is the two-body decay $b \to s\gamma$. Let us define

$$x = \frac{p_t^2}{m_b^2}, \tag{15}$$

where p_t is the transverse momentum of the strange quark with respect to the photon direction, fixed as z-axis, and m_b is the mass of the heavy quark, to be identified with the hard scale of the process[3].
In lowest order the transverse momentum distribution then is

$$\frac{d\Gamma}{dx} = \Gamma_0\ \delta(x), \tag{16}$$

that is the strange quark and the photon are emitted in opposite directions, because of momentum conservation. Acollinearity is generated by gluon emission; in $b \to s\gamma g$, i.e. at $O(\alpha_S)$, $p_t = -k_t$ while in $b \to s\gamma g_1 \dots g_n$, i.e. in higher orders, $p_t = -k_{t1} \dots - k_{tn}$.
Beside the differential distribution the partially integrated distribution[4] is also of interest

$$D(x) = \int_0^x dx'\ \frac{1}{\Gamma_0}\ \frac{d\Gamma}{dx'}. \tag{17}$$

Even though $\alpha_S(m_B)$ is small enough to justify a perturbative approach, the combination $\alpha_S^n(m_B)\log^k x$, with $0 \le k \le 2n$ can be large. A resummation, to any order in α_S, of logarithms of the same magnitude is required to obtain sensible physical results.
A partial resummation of large logarithms with next-to-leading accuracy has been performed in [2]: here we complete the calculation.
It is well known, [12], that the resummation of large logarithms is accomplished by an expression of the form:

$$D(x) = K(\alpha_S)\Sigma(x;\alpha_S) + R(x;\alpha_S), \tag{18}$$

where

[1] Γ_0 contains in principle $m^2_{b,\overline{MS}}\,(\mu_b)$ since μ_b is the renormalization point of $\hat{\mathcal{O}}_7$. As is well known, the renormalization point is arbitrary: we decided to fix it to m_b in the running mass, as usually done in the literature.
[2] The operator $\hat{\mathcal{O}}_8$ is affected by QED infrared divergences which are not relevant to our problem.
[3] Let us note that $0 \le x \le 1/4$.
[4] Since we divide the spectrum by the lowest-order rate Γ_0, we have that $D(x = 1/4) = \frac{\Gamma_{TOT}}{\Gamma_0} = 1 + O(\alpha_S)$.

- $\Sigma(x; \alpha_S)$ is a universal, process-independent, function resumming the infrared logarithms in exponentiated form. It can be expanded in a series of functions as:
$$\log \Sigma(x; \alpha_S) = L g_1(\alpha_S L) + g_2(\alpha_S L) + \alpha_S g_3(\alpha_S L) + \dots \ , \tag{19}$$
where $L = \log x$ (in general L is a large infrared logarithm). The functions g_i have a power expansion of the form
$$g_i(z) = \sum_{k=0}^{\infty} g_{i,k} \ z^k \tag{20}$$
and resum logarithms of the same size: in particular g_1 resums leading logarithms of the form $\alpha_S^n \ L^{n+1}$ and g_2 the next-to-leading ones $\alpha_S^n \ L^n$. The explicit form of $\Sigma(x; \alpha_S)$ can be found in Ref.[2];

- $K(\alpha_S)$ is a short-distance coefficient function, a process-dependent function, which can be calculated in perturbation theory:
$$K(\alpha_S) = 1 + \frac{\alpha_S C_F}{\pi} k_1 + O\left(\alpha_S^2\right). \tag{21}$$

- $R(x; \alpha_S)$ is the remainder function and satisfies the condition
$$R(x; \alpha_S) \to 0 \quad \text{for} \quad x \to 0. \tag{22}$$
It is process dependent, takes into account hard contributions and is calculable as an ordinary α_S expansion:
$$R(x; \alpha_S) = \frac{\alpha_S C_F}{\pi} r_1(x) + O\left(\alpha_S^2\right). \tag{23}$$

The result is an improved perturbative distribution, reliable in the semi-inclusive region [12], that is for small values of x, which can be matched with a fixed-order spectrum, describing the distribution for large values of x [13]. The description of the tools used to perform the resummation of infrared logarithms is far from the purpose of this note, and we refer the reader to the references [8] – [12].

In the next sections the full order α_S corrections for the coefficient function $K(\alpha_S)$ and for the remainder function $R(x; \alpha_S)$ will be explicitely calculated.

4 $O(\alpha_S)$ corrections to p_t distribution

In this section radiative corrections to the transverse momentum distribution will be calculated evaluating the Feynman diagrams depicted in fig. 2 for real gluon emissions and in fig. 3 for virtual emissions. We use the Feynman gauge where the gluon propagator is
$$D_{\mu\nu}(k) = \ -g_{\mu\nu} \frac{i}{k^2 + i\epsilon}. \tag{24}$$
The calculation is performed in dimensional regularization (DR) with the dimension of the space-time
$$n = 4 + \epsilon.$$
The operator $\hat{\mathcal{O}}_7$ from the basis (13) is inserted in the hard vertex, as discussed in section 2. Let us now define the kinematical variables: P^μ is the heavy quark momentum, p^μ the light quark momentum, k^μ the gluon momentum and q^μ the photon momentum: for real diagrams it holds that $k^2 = p^2 = q^2 = 0$ and $P^2 = m_b^2$, while, for virtual diagrams, $k^2 \neq 0$. The calculation is performed in the b rest frame, where
$$\begin{aligned} P^\mu &= (m_b, \vec{0}) \\ q^\mu &= (E_\gamma, 0, 0, E_\gamma). \end{aligned} \tag{25}$$

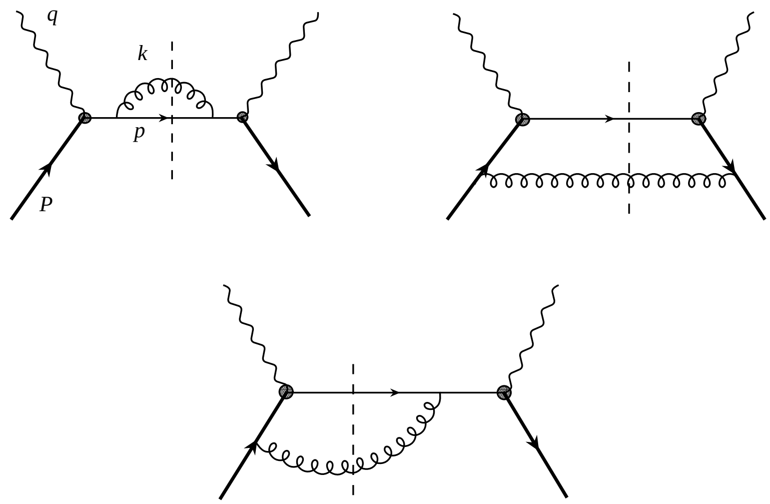

Figure 2: Real diagrams

4.1 Real diagrams

A straightforward evaluation of the diagrams in fig. 2 gives a contribution to the rate:

$$\frac{d\Gamma}{\Gamma_0} = \frac{M(\omega,t;\epsilon)}{\omega^{1-\epsilon}t^{1-\epsilon/2}}\ dt\ d\omega = \left[\frac{A_1(\omega,t;\epsilon)}{\omega^{1-\epsilon}t^{1-\epsilon/2}} + \frac{S_1(t;\epsilon)}{\omega^{1-\epsilon}} + \frac{C_1(\omega;\epsilon)}{t^{1-\epsilon/2}} + F_1(\omega,t;\epsilon)\right]\ dt\ d\omega \tag{26}$$

where

$$\begin{aligned}
A_1 &\equiv M(0,0;\epsilon) \\
S_1\,(t) &\equiv \frac{M(0,t;\epsilon)-M(0,0;\epsilon)}{t^{1-\epsilon/2}} \\
C_1\,(\omega) &\equiv \frac{M(\omega,0;\epsilon)-M(0,0;\epsilon)}{\omega^{1-\epsilon}} \\
F_1\,(\omega,t;\epsilon) &\equiv \frac{M(\omega,t;\epsilon)-M(0,t;\epsilon)-M(\omega,0;\epsilon)+M(0,0;\epsilon)}{\omega^{1-\epsilon}\,t^{1-\epsilon/2}}
\end{aligned} \tag{27}$$

and

$$\omega = \frac{2P\cdot k}{m_b^2} = \frac{2E_g}{m_b}, \qquad t = \frac{1-\cos\theta}{2},$$

with θ the angle between the gluon and the direction $-\hat{z}$ [5].
It follows from their definition that the functions $A_1(\omega,t;\epsilon)$, $S_1(\omega;\epsilon)$, $C_1(t;\epsilon)$ and $F_1(\omega,t;\epsilon)$ are finite in the soft and the collinear limit, defined respectively as

$$\omega \to 0 \quad \text{and} \quad t \to 0. \tag{28}$$

The rate in eq. (27) has to be integrated over the whole phase space with the kinematical constraint

$$\delta[x-\omega^2 t(1-t)], \tag{29}$$

[5] Let us remember that the direction $+\hat{z}$ is fixed by the photon space momentum.

which selects gluons with transverse momentum $p_t^2 = x m_b^2$ [6]. The cumulative distribution takes a contribution of the form:

$$D_R(x) = \int_0^x dx' \int_0^1 d\omega \int_0^1 dt \, \frac{1}{\Gamma_0} \, \frac{d\Gamma}{dx'}(\omega, t; \epsilon) \; \delta[x' - \omega^2 t(1-t)]. \tag{30}$$

After the integration we expect four kinds of terms:

- Poles in the regulator ϵ: they parametrize the infrared singularities and cancel in the sum with virtual diagrams because the distribution we are dealing with is infrared-safe[7];
- Logarithmic terms diverging for $x \to 0$;
- Constant terms: they enter the coefficient function $K(\alpha_S)$;
- Remainder functions: terms that vanish in the limit $x \to 0$.

Integrating over x' we have:

$$D_R(x) = \int_0^1 d\omega \int_0^1 dt \, \frac{1}{\Gamma_0} \, \frac{d\Gamma}{dx'}(\omega, t; \epsilon) \; \theta[x - \omega^2 t(1-t)]. \tag{31}$$

The remaining integrations are non-trivial because of the simultaneous presence of the kinematical constraint and by the dimensional regularization parameter ϵ. By using the identity

$$\theta[x - \omega^2 t(1-t)] \; = \; 1 - \theta[\omega^2 t(1-t) - x], \tag{32}$$

we separate these two effects and rewrite the distribution $D_R(x)$ as a difference between an integral over the whole phase space and a integral over the complementary region:

$$D_R(x) = \int_0^1 d\omega \int_0^1 dt \frac{1}{\Gamma_0} \, \frac{d\Gamma}{dx}(\omega, t; \epsilon) \; - \; \int_0^1 d\omega \int_0^1 dt \frac{1}{\Gamma_0} \, \frac{d\Gamma}{dx}(\omega, t; 0) \; \theta[\omega^2 t(1-t) - x] \; + \; O(\epsilon). \tag{33}$$

The first integral must be evaluated for $\epsilon \neq 0$ because it contains poles in ϵ, but is done over a very simple domain, independent of x. The second integral does not contain any pole in ϵ and therefore one can take the limit $\epsilon \to 0$ in the integrand. It depends on the kinematics of the process and can be integrated by introducing a suitable basis of harmonic polylogarithms as in [14]. The most convenient basis we found consists of the basic functions

$$\begin{aligned} g[0; y] \; &\equiv \; \frac{1}{y} \\ g[-1; y] \; &\equiv \; \frac{1}{y+1} \\ g[-2; y] \; &\equiv \; \frac{1}{\sqrt{y}(1+y)} \\ g[-3; y] \; &\equiv \; -\frac{\sqrt{x}}{2(1-\sqrt{x}\sqrt{y})\sqrt{y}}. \end{aligned} \tag{34}$$

The harmonic polylogarithms (HPL) of weight 1 are defined as:

$$\begin{aligned} J[a; y] \; &\equiv \; \int_0^y dy' \; g(a; y') \quad \text{for a} \neq 0 \\ J[0; y] \; &\equiv \; \log y. \end{aligned} \tag{35}$$

[6]Let us recall that for the single gluon emission $x \equiv p_t^2/m_b^2 = k_t^2/m_b^2$.

[7]A distribution is infrared-safe if it is insensitive to the emission of a soft and a collinear gluon [13].

In terms of usual functions, they read:

$$
\begin{aligned}
J[-1;y] &\equiv \log(1+y) \\
J[-2;y] &\equiv 2\arctan\sqrt{y} \\
J[-3;y] &\equiv \log(1-\sqrt{x}\sqrt{y})
\end{aligned}
\tag{36}
$$

The HPL's of weight 2 are defined for $(u,v)\neq(0,0)$ as

$$
J[u,v;y] \equiv \int_0^y dy'\; g[u;y'] \int_0^{y'} dy'' g[v;y''] \tag{37}
$$

and $J[0,0;y]=1/2\log^2 y$. HPL s of higher weight may be defined in an analogous way. They will not be used here.
The final result for real diagrams turns out to be

$$
D_R(x) = C_F \frac{\alpha_S}{\pi} \left(\frac{m_b^2}{4\pi\mu^2}\right)^{\epsilon/2} \frac{1}{\Gamma(1+\epsilon/2)} \left[\frac{2}{\epsilon^2} - \frac{5}{2\epsilon} - \frac{1}{4}\log^2 x - \frac{5}{4}\log x + \frac{1}{4} + d(x)\right], \tag{38}
$$

where $d(x)$ is a function vanishing for $x\to 0$.
The matrix elements of the remaining operators $\hat{\mathcal{O}}_i$ $(i\neq 7)$ do not contain infrared divergences. Therefore their contributions to D_R do not involve (infrared) poles in ϵ, logarithms of x and constants, but only new functions, which vanish in the limit $x\to 0$.

4.2 Virtual diagrams

Virtual corrections to $b\to s\gamma$ have been calculated in [15, 17] for a massive strange quark; we present here the computation in the massless case. The diagrams consist of self-energy corrections to the heavy and light lines and of vertex corrections to the operator $\hat{\mathcal{O}}_7$ (see fig. 3); we compute them in the $\overline{MS}$ scheme so as to be consistent with the (known) coefficient functions C_i. The computation can be done with standard Feynman parameter technique or by a reduction using the integration by part identities [18]. Let us briefly describe the evaluation of the vertex correction within the latter

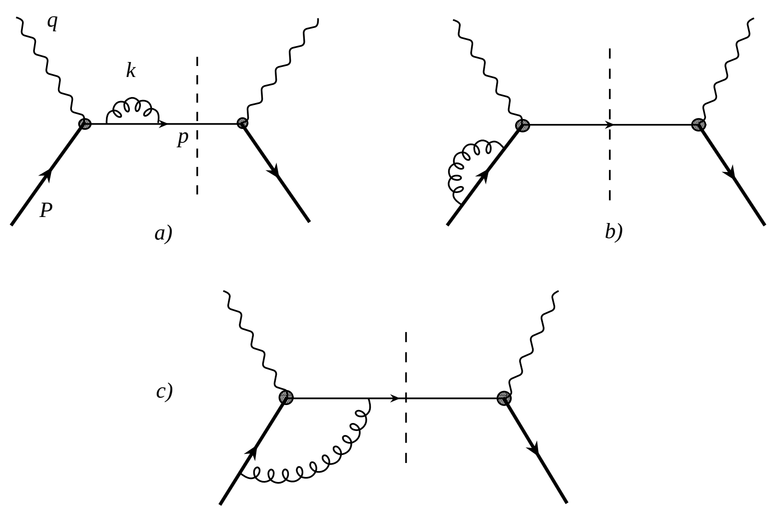

Figure 3: Virtual diagrams

method. One has to compute the scalar integral:

$$\mathcal{V} = \int \frac{d^n k}{(2\pi)^n} \, \frac{N(k^2, P\cdot k, p\cdot k; \epsilon)}{k^2[(k-P)^2 - m^2](k-p)^2}, \tag{39}$$

where

$$N(k^2, P\cdot k, p\cdot k; \epsilon) = 32 P\cdot k - 32 p\cdot k - 16 m_b^2 + \mathcal{O}(\epsilon^2) k^2 + \mathcal{O}(\epsilon^2) P\cdot k p\cdot k + \mathcal{O}(\epsilon^2)(p\cdot k)^2. \tag{40}$$

$\mathcal{V}$ has at most a double pole in ϵ coming from the product of the soft and the collinear singularities. The terms in the numerator N, which vanish in the soft limit $k_\mu \to 0$, do not give rise to soft singularities and therefore produce at most a simple pole coming from the collinear or the ultraviolet region. Therefore the $\mathcal{O}(\epsilon^2)$ terms in N do not contribute in the limit $\epsilon \to 0$.
By expressing the scalar products in the numerator as linear combinations of the denominators as

$$\begin{aligned} k\cdot p &= \frac{1}{2}\left(k^2 - (k-p)^2\right), \\ k\cdot P &= \frac{1}{2}\left(k^2 - (k-P)^2 + m_b^2\right), \end{aligned} \tag{41}$$

we can reduce $\mathcal{V}$ to a superposition of scalar integrals of the form:

$$\mathrm{T}[a,b,c] = \int \frac{d^n k}{(2\pi)^n} \, \frac{1}{[k^2]^a \, [(k-P)^2 - m_b^2]^b \, [(k-p)^2]^c} \tag{42}$$

with $a, b, c \le 1$. The above amplitudes can be related to each other by identities of the form [18]:

$$\int d^n k \, \frac{\partial}{\partial k^\mu} \, \frac{v^\mu}{[k^2]^a [(k-P)^2 - m^2]^b [(k-p)^2]^c} = 0 \tag{43}$$

with $v^\mu = k^\mu, p^\mu, P^\mu$. By explicitly evaluating the derivatives and re-expressing the scalar products using eqs. (41), one obtains relations among amplitudes with shifted indices.
By solving the above identities, one can reduce all the amplitudes to the tadpole, and one obtains for our integral[8]:

$$\mathcal{V} = \left(-\frac{16}{\epsilon} + 8 - 8\epsilon\right) \frac{1}{m_b^2} \, \mathrm{T}[0,1,0], \tag{44}$$

where

$$\mathrm{T}[0,1,0] = C_F \frac{\alpha_S}{4\pi} \left(\frac{m^2}{4\pi\mu^2}\right)^{\epsilon/2} \frac{\Gamma(-\epsilon/2)}{1+\epsilon/2} \, m_b^2. \tag{45}$$

Summing self-energies and vertex corrections, and subtracting the $1/\epsilon$ poles according to the $\overline{MS}$ scheme, one obtains for their contribution to the rate D_V [9]:

$$D_V = C_F \frac{\alpha_S}{\pi} \left(\frac{m_b^2}{4\pi\mu^2}\right)^{\epsilon/2} \Gamma\left(1 - \frac{\epsilon}{2}\right) \left[-\frac{2}{\epsilon^2} + \frac{5}{2\epsilon} + 4\log\frac{m_b}{\mu_b} - 3\right]. \tag{46}$$

We have kept the factor in front of the square bracket unexpanded to simplify the computation of the total rate.
The virtual corrections to the remaining operators $\hat{\mathcal{O}}_{i\neq 7}$ contain only (simple) ultraviolet poles in ϵ, which are removed by renormalization; their contributions to D_V amount only to finite constants and $\log m_b/\mu_b$.

[8] Such a strong reduction of 3-point function to a vacuum amplitude is possible because the only scale in the process is the heavy quark mass m_b. Virtual corrections have indeed the lowest-order kinematics $P^2 = m_b^2,\ P\cdot p = m_b^2/2,\ p^2 = q^2 = 0$.

[9] To factorize Γ_0 one has to replace $m_{b,\overline{MS}}(\mu_b)$ by $m_{b,\overline{MS}}(m_b)$ using the formula $m_{b,\overline{MS}}(\mu_b) = m_{b,\overline{MS}}(m_b)(1 + \frac{3}{2}\frac{C_F\alpha_S}{\pi})$.

4.3 Final result

Summing real and virtual contributions, the transverse momentum distribution for the decay $b \to s\gamma$ reads, to $O(\alpha_S)$:

$$D(x) = 1 \ + \ C_F \frac{\alpha_S}{\pi} \ \left[-\frac{1}{4} \ \log^2 x - \ \frac{5}{4} \ \log x + f + \ d(x) \right]. \tag{47}$$

As expected, the result contains a double logarithm and a single logarithm of x, a finite term f and a function $d(x)$ vanishing in the limit $x \to 0$.
By expanding the resummed formula to order α_S one obtains:

$$\begin{aligned} D(x) \ &= \ \left(1 + \frac{C_F \alpha_S}{\pi} k_1\right) \left(1 - \frac{A_1}{4} \alpha_S \log^2 x + B_1 \alpha_S \log x\right) + \frac{C_F \alpha_S}{\pi} r(x) \\ &= \ 1 - \frac{A_1}{4} \alpha_S \log^2 x + B_1 \alpha_S \log x + \frac{C_F \alpha_S}{\pi} k_1 + \frac{C_F \alpha_S}{\pi} r(x) + O(\alpha_S^2). \end{aligned} \tag{48}$$

By identifying the resummed result expanded to $O(\alpha_S)$ with the fixed-order one — matching procedure — we check the values for A_1 and B_1 evaluated in our previous paper using general properties of QCD radiation [2] and we extract the value of the coefficient function:

$$k_1 \ = \ f \ = \ - \ \frac{11}{4} - \ \frac{\pi^2}{12} + \ 4 \log \frac{m_b}{\mu}, \tag{49}$$

as well as the remainder function $r_1(x) = d(x)$. As explained in previous sections, the remaining operators $\hat{\mathcal{O}}_{i \neq 7}$ contribute to $D(x)$ only by finite terms $\tilde{r}_i$ and remainder functions. Since the constants $\tilde{r}_i$ come from virtual diagrams alone, we can quote their result from [15] and present an improved formula for the coefficient function, in analogy with [16]:

$$K(\alpha_S) \ = \ 1 + \frac{\alpha_S}{2\pi} \sum_{i=1}^{8} \frac{C_i^{(0)}(\mu_b)}{C_7^{(0)}(\mu_b)} \left(\Re \ \tilde{r}_i + \gamma_{i7}^{(0)} \log \frac{m_b}{\mu_b} \right) + \frac{\alpha_S}{2\pi} \frac{C_7^{(1)}(\mu_b)}{C_7^{(0)}(\mu_b)} + \mathcal{O}(\alpha_S^2) \tag{50}$$

where

$$\begin{aligned} \tilde{r}_i \ &= \ r_i \qquad i \neq 7 \\ \tilde{r}_7 \ &= \ \frac{8}{3} \left(f - 4 \log \frac{m_b}{\mu_b} \right) = -\frac{22}{3} - \frac{2\pi^2}{9}. \end{aligned} \tag{51}$$

Let us remark that only the coefficients related to the operators with $i = 1, 2, 7, 8$ are relevant, because the others are multiplied by very small coefficient functions and can be neglected:

$$\begin{aligned} r_1 \ &= \ -\frac{1}{6} r_2 \\ \Re \ r_2 \ &= \ -4.092 - 12.78(0.29 - m_c/m_b) \\ r_8 \ &= \ \frac{4}{27}(33 - 2\pi^2). \end{aligned} \tag{52}$$

The analytic expressions for the coefficient functions as well as a standard numerical evalutation are given in [7]. The anomalous dimension $\gamma_{77}^{(0)}$ is derived from the coefficient of the logarithmic term in k_1. The values of $\gamma_{i7}^{(0)}$ are [7]:

$$\gamma_{i7}^{(0)} = \left(-\frac{208}{243}, \frac{416}{81}, -\frac{176}{81}, -\frac{152}{243}, -\frac{6272}{81}, \frac{4624}{243}, \frac{32}{3}, -\frac{32}{9} \right). \tag{53}$$

Equation(50) is the main result of our paper and allows a complete resummation to NLO of transverse

momentum logarithms.
The explicit calculation of the remainder function in (48) reads

$$
\begin{aligned}
r(\tau) &= \frac{(\tau-1)(49\tau^8+468\tau^7+1797\tau^6+3642\tau^5+4450\tau^4+3642\tau^3+1797\tau^2+468\tau+49)}{12(\tau+1)^5(\tau^2+3\tau+1)^2} \\
&+ \frac{-5-61\tau-317\tau^2-912\tau^3-1622\tau^4-1934\tau^5-1622\tau^6-912\tau^7-317\tau^8-61\tau^9-5\tau^{10}}{4(\tau+1)^6(\tau^2+3\tau+1)^2}\log\tau \\
&- J[0,-3,\tau]+J[0,-3,1/\tau]-2J[0,-1,\tau]+J[-1,0,\tau]+J[-1,-3,\tau]-J[-1,-3,1/\tau] \\
&- 2\sqrt{\tau}\arctan(\sqrt{\tau})\frac{(\tau+1)(2\tau^2+7\tau+2)}{(\tau^2+3\tau+1)^2}+\frac{\pi}{2}\sqrt{\tau}\frac{(\tau+1)(2\tau^2+7\tau+2)}{(\tau^2+3\tau+1)^2}+\frac{49}{12} \\
&+ \frac{5}{4}\log\tau-\frac{5}{2}\log(\tau+1)+\log^2(\tau+1), \qquad (54)
\end{aligned}
$$

where

$$\tau = \frac{1-\sqrt{1-4x}}{1+\sqrt{1-4x}}. \qquad (55)$$

Let us notice that τ behaves as x for small values of the transverse momentum

$$\tau(x) = x + O(x^2) \qquad (56)$$

and it is a unitary variable

$$
\begin{array}{lll}
\tau \to 0 & \text{for} & x \to 0 \\
\tau \to 1 & \text{for} & x \to 1/4.
\end{array}
$$

The relation (55) may be inverted as

$$x = \frac{\tau}{(\tau+1)^2}. \qquad (57)$$

One can easily check that $r(\tau)$ vanishes for $\tau \sim x \to 0$, by using the properties

$$J[0,-1,0] = J[0,-3,0] = J[-1,0,0] = J[-1,-3,0] = 0 \qquad (58)$$

$$\lim_{\tau\to 0} J[0,-3,1/\tau] = \lim_{\tau\to 0} J[-1,-3,1/\tau] = -\frac{\pi^2}{3}. \qquad (59)$$

5 Conclusions

To sum up: eq. (47), which contains the final result, represents the full evaluation to $O(\alpha_S)$ of the transverse momentum distribution. It is explicitly given in terms of an analytic expression.
Contrary to what happens in hard processes at much larger energies, at the energy scales involved here for the b decay the remainder function contribution does play a more important role.
A straightforward numerical evaluation of the remainder function $r(x)$ of eq. (48) allows us to conclude that its contribution can be safely neglected for small values of x, up to $x \simeq 0.1$, where it approaches the zero limit of $x \to 0$. For larger values of x, however, the size of its contribution increases to reach values of the order of the 10–15% of the combined leading and next-to-leading logarithmic terms.
A detailed report describing the calculation giving rise the $O(\alpha_S)$ evaluation presented here, together with an analysis of the related phenomenological impact, will be presented in a future article [3].

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